System's Half Width

Introduction

This paper is going to discuss about the system response for the measurement device development in which the author has ever engaged for a long time. The response means a degradation, a blur. The author remembers that handling the time series is a lot, but it is not limited for only the time series. Now, the theme is how to estimate the blur, rather than the numerical calculation method.

Half Width

Half width at half maximum h is a characteristic to measure the blur(**Fig.1**).

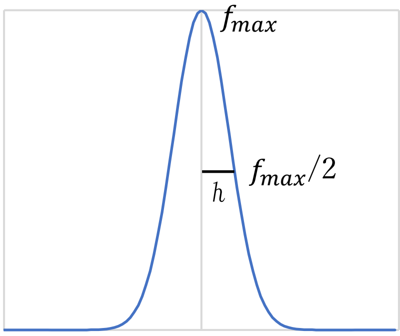

Fig.1 Half Width at Half Maximum

The half width at half maximum is defined by the width h at the position that the wave's height becomes a half of its maximum. For short, the half width, may mean the full width at half maximum, generally. This paper says the half width at half maximum as the half width because of the comparison to σ.

Blur(Signal Degradation)

Generally, input the source signal f, then the output, the observed signal m will be got (**Fig.2**).

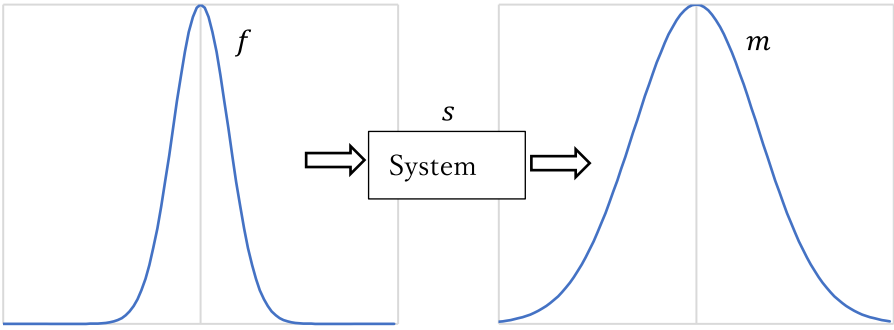

Fig.2 Blur(Signal Degradation)

As mathematics, the time domain signal is expressed by convoluting the system response function(1).

$$m(t) = f(t) \otimes s(t) = \int_{-\infty}^{\infty} f(\tau) s(t-\tau)\, d\tau \quad (1)$$

$\otimes$: Convolution

In the case of time series, it becomes a superposition(addition) of the impulse response. This might be expressed by the semi-infinite integral interval.

Gaussian Function's Assumption

At first, let's think about the case that both the source signal and system response function are gaussian functions. Before starting, clear the relation between σ and the half width. The gaussian function is (2).

$$\exp\left(\frac{-(x-\bar{x})^2}{2\sigma^2}\right) \quad (2)$$

As the half width h, the relation to σ will be got by (3).

$$\frac{1}{2} = \exp\left(\frac{-\left((\bar{x}+h)-\bar{x}\right)^2}{2\sigma^2}\right) \quad (3)$$

Assume $\bar{x} = 0$ for short(4), it becomes (5).

$$\frac{1}{2} = \exp\left(\frac{-h^2}{2\sigma^2}\right) \quad (4)$$

$$2 = \exp\left(\frac{h^2}{2\sigma^2}\right)$$

$$\frac{h^2}{2\sigma^2} = \ln 2$$

$$h^2 = 2\sigma^2 \ln 2$$

$$h = \sqrt{2\ln 2}\,\sigma$$

$$h \approx 1.18\sigma \quad (5)$$

The half width is proportional to the deviation. Here, define the source signal and the system response function as (6), (7).

$$f(t) = a_f \exp\left(\frac{-t^2}{2{\sigma_f}^2}\right) \quad (6)$$

$$s(t) = a_s \exp\left(\frac{-t^2}{2{\sigma_s}^2}\right) \quad (7)$$

Convolute, it becomes the observed signal $m(t)$(8)→(9).

$$m(t) = f(t) \otimes s(t) = \int_{-\infty}^{\infty} f(\tau)s(t-\tau)\,d\tau \quad (8)$$

$$= \int_{-\infty}^{\infty} a_f \exp\left(\frac{-\tau^2}{2{\sigma_f}^2}\right) a_s \exp\left(\frac{-(t-\tau)^2}{2{\sigma_s}^2}\right) d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(\frac{-\tau^2}{2{\sigma_f}^2} + \frac{-(t-\tau)^2}{2{\sigma_s}^2}\right) d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(\frac{-{\sigma_s}^2\tau^2 - {\sigma_f}^2(t^2 - 2\tau t + \tau^2)}{2{\sigma_f}^2{\sigma_s}^2}\right) d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2{\sigma_f}^2{\sigma_s}^2}\left(\left({\sigma_s}^2 + {\sigma_f}^2\right)\tau^2 - 2{\sigma_f}^2 t\tau + {\sigma_f}^2 t^2\right)\right) d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2{\sigma_f}^2{\sigma_s}^2}\left(\left(\sqrt{{\sigma_s}^2 + {\sigma_f}^2}\tau - \frac{{\sigma_f}^2 t}{\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}\right)^2 - \frac{{\sigma_f}^4 t^2}{{\sigma_s}^2 + {\sigma_f}^2} + {\sigma_f}^2 t^2\right)\right) d\tau$$

$$= a_f a_s \exp\left(\frac{{\sigma_f}^4 t^2 - {\sigma_f}^2 t^2\left({\sigma_s}^2 + {\sigma_f}^2\right)}{2{\sigma_f}^2{\sigma_s}^2\left({\sigma_s}^2 + {\sigma_f}^2\right)}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2{\sigma_f}^2{\sigma_s}^2}\left(\sqrt{{\sigma_s}^2 + {\sigma_f}^2}\tau - \frac{{\sigma_f}^2 t}{\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}\right)^2\right) d\tau$$

$$= a_f a_s \exp\left(\frac{-{\sigma_f}^2{\sigma_s}^2 t^2}{2{\sigma_f}^2{\sigma_s}^2\left({\sigma_s}^2 + {\sigma_f}^2\right)}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2{\sigma_f}^2{\sigma_s}^2}\left(\frac{\left({\sigma_s}^2 + {\sigma_f}^2\right)\tau - {\sigma_f}^2 t}{\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}\right)^2\right) d\tau$$

$$= a_f a_s \exp\left(\frac{-t^2}{2\left({\sigma_s}^2 + {\sigma_f}^2\right)}\right) \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\left({\sigma_s}^2 + {\sigma_f}^2\right)\tau - {\sigma_f}^2 t}{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}\right)^2\right) d\tau \quad (9)$$

Put (10) to the integral term, the permutation integral is possible.

$$x = \frac{\left({\sigma_s}^2 + {\sigma_f}^2\right)\tau - {\sigma_f}^2 t}{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}} \quad (10)$$

$$\left({\sigma_s}^2 + {\sigma_f}^2\right)\tau - {\sigma_f}^2 t = \sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}x$$

$$\tau = \frac{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}x + {\sigma_f}^2 t}{{\sigma_s}^2 + {\sigma_f}^2}$$

$$\frac{d\tau}{dx} = \frac{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}{{\sigma_s}^2 + {\sigma_f}^2}$$

Therefore, the integral term becomes (11).

$$\int_{-\infty}^{\infty} \exp\left(-\left(\frac{\left({\sigma_s}^2 + {\sigma_f}^2\right)\tau - {\sigma_f}^2 t}{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}\right)^2\right) d\tau$$

$$= \int_{-\infty}^{\infty} \exp(-x^2) \frac{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}{{\sigma_s}^2 + {\sigma_f}^2} dx = \frac{\sqrt{2}\sigma_f\sigma_s\sqrt{{\sigma_s}^2 + {\sigma_f}^2}}{{\sigma_s}^2 + {\sigma_f}^2} \int_{-\infty}^{\infty} \exp(-x^2)\, dx \quad (11)$$

Gaussian integral formula is (12).

$$\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi} \quad (12)$$

Above all, the observed signal becomes (13).

$$m(t) = a_f a_s \exp\left(\frac{-t^2}{2\left({\sigma_s}^2 + {\sigma_f}^2\right)}\right) \frac{\sqrt{2}\sigma_f \sigma_s \sqrt{{\sigma_s}^2 + {\sigma_f}^2}}{{\sigma_s}^2 + {\sigma_f}^2} \sqrt{\pi}$$

$$= a_f a_s \sigma_f \sigma_s \sqrt{\frac{2\pi}{{\sigma_s}^2 + {\sigma_f}^2}} \exp\left(\frac{-t^2}{2\left({\sigma_s}^2 + {\sigma_f}^2\right)}\right) \quad (13)$$

The gaussian function's convolution combination becomes the gaussian function. Then, the observed signal's deviation σ_m is (14)..

$$\sigma_m = \sqrt{{\sigma_s}^2 + {\sigma_f}^2} \quad (14)$$

Therefore, the half width is same(15).

$$h_m = \sqrt{{h_s}^2 + {h_f}^2} \quad (15)$$

How about the case that the integral interval is the semi-infinite? The gaussian integral is (16).

$$\int_0^\infty \exp(-x^2)\, dx = \frac{1}{2}\sqrt{\pi} \quad (16)$$

Therefore, although the integral interval has changed, the deviation still remains (14).

Lorentzian Function's Assumption

The lorentzian function is (17).

$$\frac{\sigma}{(x-\bar{x})^2+\sigma^2} \quad (17)$$

It is a similar shape to the gaussian function. Let's estimate the half width just like the gaussian function(18).

$$\frac{1}{2}\cdot\frac{1}{\sigma}=\frac{\sigma}{h^2+\sigma^2}$$

$$h^2+\sigma^2=2\sigma^2$$

$$h^2=\sigma^2$$

$$h=\sigma \quad (18)$$

The lorentzian function's **half width is the deviation itself.** Here, define the source signal and the system response function as (19), (20).

$$f(t)=a_f\frac{\sigma_f}{t^2+{\sigma_f}^2} \quad (19)$$

$$s(t)=a_s\frac{\sigma_f}{t^2+{\sigma_s}^2} \quad (20)$$

Convolute, it becomes the observed signal $m(t)$(21).

$$m(t)=f(t)\otimes s(t) \quad (21)$$

$$=\int_{-\infty}^{\infty}a_f\frac{\sigma_f}{\tau^2+{\sigma_f}^2}a_s\frac{\sigma_s}{(t-\tau)^2+{\sigma_s}^2}d\tau$$

$$=a_fa_s\sigma_f\sigma_s\int_{-\infty}^{\infty}\frac{1}{\left(\tau^2+{\sigma_f}^2\right)\left((\tau-t)^2+{\sigma_s}^2\right)}d\tau$$

The format is the rational function(22).

$$\frac{1}{\left(\tau^2+{\sigma_f}^2\right)\left((\tau-t)^2+{\sigma_s}^2\right)}=\frac{A\tau+B}{\tau^2+{\sigma_f}^2}+\frac{C\tau+D}{(\tau-t)^2+{\sigma_s}^2} \quad (22)$$

$$(A\tau+B)\left((\tau-t)^2+{\sigma_s}^2\right)+(C\tau+D)\left(\tau^2+{\sigma_f}^2\right)=1$$

$$(A\tau+B)\left(\tau^2-2\tau t+t^2+{\sigma_s}^2\right)+(C\tau+D)\left(\tau^2+{\sigma_f}^2\right)-1=0$$

$$(A+C)\tau^3+(-2tA+B+D)\tau^2+\left(\left(t^2+{\sigma_s}^2\right)A-2tB+{\sigma_f}^2C\right)\tau+\left(t^2+{\sigma_s}^2\right)B+{\sigma_f}^2D-1=0$$

Therefore, the coefficients' coalition is (23)～(26).

$$A + C = 0 \quad (23)$$
$$-2tA + B + D = 0 \quad (24)$$
$$(t^2 + {\sigma_s}^2)A - 2tB + {\sigma_p}^2 C = 0 \quad (25)$$
$$(t^2 + {\sigma_s}^2)B + {\sigma_p}^2 D - 1 = 0 \quad (26)$$

Solve the coalition(27)～(30).

$$A = -C$$

$$2tC + B + D = 0$$
$$B = -D - 2tC$$
$$-(t^2 + {\sigma_s}^2)C - 2t(-D - 2tC) + {\sigma_f}^2 C = 0$$
$$2tD = (t^2 + {\sigma_s}^2)C - 4t^2 C - {\sigma_f}^2 C$$
$$D = \frac{-3t^2 + {\sigma_s}^2 - {\sigma_f}^2}{2t} C$$

$$B = -\frac{-3t^2 + {\sigma_s}^2 - {\sigma_f}^2}{2t} C - 2tC = \frac{3t^2 - {\sigma_s}^2 + {\sigma_f}^2 - 4t^2}{2t} C$$
$$= \frac{-t^2 - {\sigma_s}^2 + {\sigma_f}^2}{2t} C$$

$$(t^2 + {\sigma_s}^2)\frac{-t^2 - {\sigma_s}^2 + {\sigma_f}^2}{2t} C + {\sigma_f}^2 \frac{-3t^2 + {\sigma_s}^2 - {\sigma_f}^2}{2t} C - 1 = 0$$
$$\frac{(t^2 + {\sigma_s}^2)(-t^2 - {\sigma_s}^2 + {\sigma_f}^2) + {\sigma_f}^2(-3t^2 + {\sigma_s}^2 - {\sigma_f}^2)}{2t} C = 1$$
$$C = \frac{2t}{-t^4 + (-{\sigma_s}^2 + {\sigma_f}^2 - {\sigma_s}^2 - 3{\sigma_f}^2)t^2 + (-{\sigma_s}^4 + {\sigma_s}^2{\sigma_f}^2 + {\sigma_f}^2{\sigma_s}^2 - {\sigma_f}^4)}$$
$$= \frac{2t}{-t^4 - 2({\sigma_s}^2 + {\sigma_f}^2)t^2 - ({\sigma_s}^4 - 2{\sigma_s}^2{\sigma_f}^2 + {\sigma_f}^4)}$$
$$= \frac{2t}{-t^4 - 2({\sigma_s}^2 + {\sigma_f}^2)t^2 - ({\sigma_s}^2 - {\sigma_f}^2)^2} \quad (27)$$

$$B = \frac{-t^2 - {\sigma_s}^2 + {\sigma_f}^2}{-t^4 - 2({\sigma_s}^2 + {\sigma_f}^2)t^2 - ({\sigma_s}^2 - {\sigma_f}^2)^2} \quad (28)$$
$$D = \frac{-3t^2 + {\sigma_s}^2 - {\sigma_f}^2}{-t^4 - 2({\sigma_s}^2 + {\sigma_f}^2)t^2 - ({\sigma_s}^2 - {\sigma_f}^2)^2} \quad (29)$$

$$A = \frac{-2t}{-t^4 - 2(\sigma_s{}^2 + \sigma_f{}^2)t^2 - (\sigma_s{}^2 - \sigma_f{}^2)^2} \quad (30)$$

Therefore, the integral becomes (31).

$$\int_{-\infty}^{\infty} \frac{1}{(\tau^2 + \sigma_f{}^2)((\tau - t)^2 + \sigma_s{}^2)} d\tau = \int_{-\infty}^{\infty} \left(\frac{A\tau + B}{\tau^2 + \sigma_f{}^2} + \frac{C\tau + D}{(\tau - t)^2 + \sigma_s{}^2} \right) d\tau$$

$$= \int_{-\infty}^{\infty} \left(\frac{A\tau}{\tau^2 + \sigma_f{}^2} + \frac{B}{\tau^2 + \sigma_f{}^2} + \frac{C\tau}{(\tau - t)^2 + \sigma_s{}^2} + \frac{D}{(\tau - t)^2 + \sigma_s{}^2} \right) d\tau \quad (31)$$

$(\tau - t)$ can be the permutation integral, as the integral interval is same after the variable conversion, continue(32).

$$x = \tau - t$$

$$\frac{d\tau}{dx} = 1$$

$$= \int_{-\infty}^{\infty} \frac{A\tau}{\tau^2 + \sigma_f{}^2} d\tau + \int_{-\infty}^{\infty} \frac{B}{\tau^2 + \sigma_f{}^2} d\tau + \int_{-\infty}^{\infty} \frac{C(x + t)}{x^2 + \sigma_s{}^2} \frac{d\tau}{dx} dx + \int_{-\infty}^{\infty} \frac{D}{x^2 + \sigma_s{}^2} \frac{d\tau}{dx} dx$$

$$= \int_{-\infty}^{\infty} \frac{A\tau}{\tau^2 + \sigma_f{}^2} d\tau + \int_{-\infty}^{\infty} \frac{B}{\tau^2 + \sigma_f{}^2} d\tau + \int_{-\infty}^{\infty} \frac{Cx}{x^2 + \sigma_s{}^2} dx + \int_{-\infty}^{\infty} \frac{Ct}{x^2 + \sigma_s{}^2} dx + \int_{-\infty}^{\infty} \frac{D}{x^2 + \sigma_s{}^2} dx$$

$$= \left[\frac{A}{2} \log(\tau^2 + \sigma_f{}^2) \right]_{-\infty}^{\infty} + \left[\frac{B}{\sigma_f} \tan^{-1} \frac{\tau}{\sigma_f} \right]_{-\infty}^{\infty} + \left[\frac{C}{2} \log(x^2 + \sigma_s{}^2) \right]_{-\infty}^{\infty} + \left[\frac{Ct}{\sigma_s} \tan^{-1} \frac{x}{\sigma_s} \right]_{-\infty}^{\infty}$$
$$+ \left[\frac{D}{\sigma_s} \tan^{-1} \frac{x}{\sigma_s} \right]_{-\infty}^{\infty}$$

$$= \frac{A}{2} \left(\lim_{\tau \to \infty} \log(\tau^2 + \sigma_f{}^2) - \lim_{\tau \to \infty} \log((-\tau)^2 + \sigma_f{}^2) \right) + \frac{B}{\sigma_f} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) + \frac{C}{2}(0) + \frac{Ct}{\sigma_s}(\pi) + \frac{D}{\sigma_s}(\pi)$$

$$= \pi \left(\frac{B}{\sigma_f} + \frac{Ct + D}{\sigma_s} \right) = \frac{\pi}{\sigma_f \sigma_s} \left(\sigma_s B + \sigma_f (Ct + D) \right) \quad (32)$$

Put together, the observed signal becomes (33).

$$m(t) = a_f a_s \sigma_f \sigma_s \frac{\pi}{\sigma_f \sigma_s} \left(\sigma_s B + \sigma_f (Ct + D) \right) = a_f \sigma_s \pi \left(\sigma_s B + \sigma_f (Ct + D) \right)$$

$$= a_f a_s \pi \left(\sigma_s \frac{-t^2 - \sigma_s{}^2 + \sigma_f{}^2}{-t^4 - 2(\sigma_s{}^2 + \sigma_f{}^2)t^2 - (\sigma_s{}^2 - \sigma_f{}^2)^2} + \sigma_f \frac{2t^2 - 3t^2 + \sigma_s{}^2 - \sigma_f{}^2}{-t^4 - 2(\sigma_s{}^2 + \sigma_f{}^2)t^2 - (\sigma_s{}^2 - \sigma_f{}^2)^2} \right)$$

$$= \frac{\pi}{\sigma_f \sigma_s} \left(\frac{-(\sigma_s + \sigma_f)t^2 - \sigma_s{}^3 + \sigma_s \sigma_f{}^2 + \sigma_f \sigma_s{}^2 - \sigma_f{}^3}{-t^4 - 2(\sigma_s{}^2 + \sigma_f{}^2)t^2 - (\sigma_s{}^2 - \sigma_f{}^2)^2} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)t^2 + \sigma_s{}^3 - \sigma_s \sigma_f{}^2 - \sigma_f \sigma_s{}^2 + \sigma_f{}^3}{t^4 + 2(\sigma_s{}^2 + \sigma_f{}^2)t^2 + (\sigma_s{}^2 - \sigma_f{}^2)^2} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)t^2 + (\sigma_s - \sigma_f)(\sigma_s{}^2 - \sigma_f{}^2)}{(t^2 + \sigma_s{}^2 + \sigma_f{}^2)^2 - (\sigma_s{}^2 + \sigma_f{}^2)^2 + (\sigma_s{}^2 - \sigma_f{}^2)^2} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)t^2 + (\sigma_s - \sigma_f)^2(\sigma_s + \sigma_f)}{(t^2 + \sigma_s{}^2 + \sigma_f{}^2)^2 + (\sigma_s{}^2 - \sigma_f{}^2 + \sigma_s{}^2 + \sigma_f{}^2)\left(\sigma_s{}^2 - \sigma_f{}^2 - (\sigma_s{}^2 + \sigma_f{}^2)\right)} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)\left(t^2 + (\sigma_s - \sigma_f)^2\right)}{(t^2 + \sigma_s{}^2 + \sigma_f{}^2)^2 + 2\sigma_s{}^2 \cdot -2\sigma_f{}^2} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)\left(t^2 + (\sigma_s - \sigma_f)^2\right)}{(t^2 + \sigma_s{}^2 + \sigma_f{}^2)^2 - 4\sigma_s{}^2\sigma_f{}^2} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)\left(t^2 + (\sigma_s - \sigma_f)^2\right)}{(t^2 + \sigma_s{}^2 + \sigma_f{}^2 + 2\sigma_s\sigma_f)(t^2 + \sigma_s{}^2 + \sigma_f{}^2 - 2\sigma_s\sigma_f)} \right)$$

$$= a_f a_s \pi \left(\frac{(\sigma_s + \sigma_f)\left(t^2 + (\sigma_s - \sigma_f)^2\right)}{\left(t^2 + (\sigma_s + \sigma_f)^2\right)\left(t^2 + (\sigma_s - \sigma_f)^2\right)} \right)$$

$$= a_f a_s \pi \left(\frac{\sigma_s + \sigma_f}{t^2 + (\sigma_s + \sigma_f)^2} \right) \quad (33)$$

The lorentzian function's convolution combination becomes the lorentzian function. From the above, the observed signal's deviation σ_m becomes (34).

$$\sigma_m = \sigma_s + \sigma_f \quad (34)$$

Therefore, the halfwidth is (35).

$$h_m = h_s + h_f \quad (35)$$

It is different from the gaussian function, it can be expressed by the deviation's addition. In addition, how about the case that the integral interval is the semi-infinite(36)?

$$\left[\frac{A}{2}\log\left(\tau^2+{\sigma_f}^2\right)\right]_0^{\infty}+\left[\frac{B}{\sigma_f}\tan^{-1}\frac{\tau}{\sigma_f}\right]_0^{\infty\infty}+\left[\frac{C}{2}\log(x^2+{\sigma_s}^2)\right]_0^{\infty}+\left[\frac{Ct}{\sigma_s}\tan^{-1}\frac{x}{\sigma_s}\right]_{-\infty}^{\infty}+\left[\frac{D}{\sigma_s}\tan^{-1}\frac{x}{\sigma_s}\right]_0^{\infty}$$

$$=\frac{A}{2}\left(\lim_{\tau\to\infty}\log\left(\tau^2+{\sigma_f}^2\right)-\lim_{\tau\to 0}\log\left(\tau^2+{\sigma_f}^2\right)\right)+\frac{B}{\sigma_f}\left(\frac{\pi}{2}-0\right)$$

$$+\frac{C}{2}\left(\lim_{x\to\infty}\log(x^2+{\sigma_s}^2)-\lim_{x\to 0}\log(x^2+{\sigma_s}^2)\right)+\frac{Ct}{\sigma_s}\left(\frac{\pi}{2}\right)+\frac{D}{\sigma_s}\left(\frac{\pi}{2}\right)$$

$$=\frac{A}{2}\left(\lim_{\tau\to\infty}\log\left(\tau^2+{\sigma_f}^2\right)-2\log\sigma_f\right)-\frac{A}{2}\left(\lim_{x\to\infty}\log(x^2+{\sigma_s}^2)-2\log\sigma_s\right)$$

$$+\frac{\pi}{2}\left(\frac{B}{\sigma_f}+\frac{Ct+D}{\sigma_s}\right)=A\log\sigma_s-A\log\sigma_f+\frac{\pi}{2}\left(\frac{B}{\sigma_f}+\frac{Ct+D}{\sigma_s}\right)$$

$$=A\log\left(\frac{\sigma_s}{\sigma_f}\right)+\frac{\pi}{2}\left(\frac{B}{\sigma_f}+\frac{Ct+D}{\sigma_s}\right)=A\log\left(\frac{\sigma_s}{\sigma_f}\right)+\frac{a_f a_s\pi}{2}\left(\frac{\sigma_s+\sigma_f}{t^2+\left(\sigma_s+\sigma_f\right)^2}\right)\quad(36)$$

It is difficult to derive the analytical solution. But it can be approximated by $\log\left(\sigma_s/\sigma_f\right)\approx 0$ around $\sigma_s\approx\sigma_f$. In addition, if $t(>1)\to\infty$, then $A\to 0$, A 's decreasing speed with t 's increase is more rapid than the lorentzian function's one. From above, it roughly conforms to (34), (35).

Gaussian Function and Lorentzian Function's Assumption

How about the gaussian function and Lorentzian function(37), (38)?

$$f(t) = a_f \frac{\sigma_f}{t^2 + {\sigma_f}^2} \quad (37)$$

$$s(t) = a_s \exp\left(\frac{-t^2}{2{\sigma_s}^2}\right) \quad (38)$$

Convolute, it becomes the observed signal $m(t)$(39).

$$m(t) = a_f a_s \int_{-\infty}^{\infty} \frac{\sigma_f}{\tau^2 + {\sigma_f}^2} \exp\left(\frac{-(t-\tau)^2}{2{\sigma_s}^2}\right) d\tau \quad (39)$$

It is not integrable by any means. But it has a naming called as Voigt function. Although there must be a reason for it, it is unknown for the author.

Now, although the integral is impossible, this paper's interest is the half width. It can be estimated from the above result(**Fig.3**).

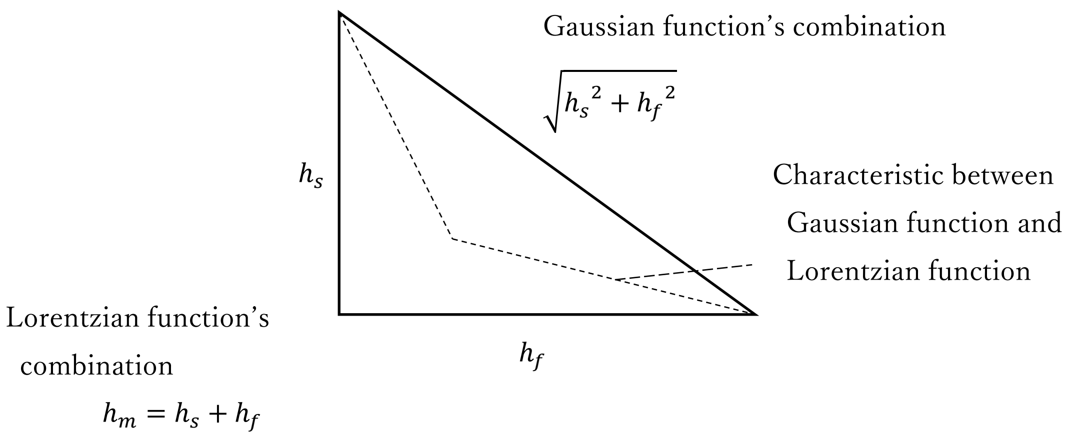

Fig.3 Half Width Relation

Once more the gaussian function's combination is (14), the lorentzian function's combination is (35).

$$h_m = \sqrt{{h_s}^2 + {h_f}^2} \quad (14)$$

$$h_m = h_s + h_f \quad (35)$$

Although, the gaussian function and lorentzian function's shape is similar, if their deviation is equal, the lorentzian function is more rapid (decay) curve than the gaussian curve.
The characteristic will be estimated to be the intermediate between them(40).

$$\sqrt{{h_s}^2 + {h_f}^2} \leq h_m \leq h_s + h_f \quad (40)$$

What is Lorentzian Function?

There are various shapes for the curve. In the case to estimate the blur, the decay's speed is important. There are such as the polynomial function, the exponent function and the gaussian function to express the decay. Then, the estimation whether the curve is similar to the gaussian function by the curve fitting can be executed.

The author thinks that the transient characteristics is absolutely representative for the exponent function. The gaussian function has an image derived from the probability. In the other hand, what kind of a function is the lorentzian function? In the optics, the function often appears. Especially, to express the spectrum's shape by the lorentzian distribution is general. As the derivative equation for the physics, they might say the resonance phenomenon. Speaking of the derivative equation, it is the model handling almost the vibration. 2 order derivative equation's format is (41).

$$\alpha \frac{d^2 f}{dt^2} + \beta \frac{df}{dt} + \gamma f = 0 \quad (41)$$

This format becomes (42) for the equation of motion, (43) for the electricity.

$$m \frac{d^2 x}{dt^2} + \mu \frac{dx}{dt} + kx \quad (42)$$

x: Space

m: Mass

μ: Friction coefficient

k: Spring constant

$$l \frac{d^2 q}{dt^2} + r \frac{dq}{dt} + \frac{1}{c} q \quad (43)$$

q: Electric charge

l: Inductor

r: Resistor

c: Capacitor

Here, normalize (41) by the second derivative coefficient, it is expressed by (44).

$$\frac{d^2f}{dt^2} + \beta\frac{df}{dt} + \gamma f = 0 \quad (44)$$

According to the author's recognition, the right side of (44) expresses an external disturbance(external force)(45).

$$\frac{d^2f}{dt^2} + \beta\frac{df}{dt} + \gamma f = ge^{i\omega t} \quad (45)$$

g: External disturbance(external force)'s amplitude

Although the general solution's derivation for (44) is easy by using Laplace transform, it is well known as the vibration(46).

$$f(t) = ae^{i\omega t} \quad (46)$$

Therefore, the external force and the system are considered to resonate together(single angular frequency). Substitute this to (45)(47).

$$-a\omega^2e^{i\omega t} + i\beta a\omega e^{i\omega t} + \gamma ae^{i\omega t} = ge^{i\omega t} \quad (47)$$

$$(-\omega^2 + i\beta\omega + \gamma)a = g$$

$$a = \frac{g}{\gamma - \omega^2 + i\beta\omega}$$

Therefore, f is (48).

$$f = \frac{g}{\gamma - \omega^2 + i\beta\omega}e^{-i\omega t} \quad (48)$$

The amplitude is (49).

$$|f| = \frac{g}{\sqrt{(\gamma - \omega^2)^2 + (\beta\omega)^2}} \quad (49)$$

The square as the power is (50).

$$|f|^2 = \frac{g^2}{(\gamma - \omega^2)^2 + (\gamma\omega)^2} \quad (50)$$

It looks like the lorentzian function. As the author can not find a good example, this may not be an example to understand easily. Aside, there is a case to encounter the lorentzian function.

Gaussian Function and Exponent Function's Assumption

Finally, the gaussian function and exponent function(decay: transient characteristics) are (51), (52).

$$f(t) = a_f \exp\left(\frac{-t^2}{2{\sigma_f}^2}\right) \quad (51)$$

$$s(t) = a_s \exp(-\lambda t) \quad (52)$$

Convolute, it becomes the observed signal $m(t)$(53).

$$m(t) = a_f a_s \int_{-\infty}^{\infty} \exp\left(\frac{-\tau^2}{2{\sigma_f}^2}\right) \exp\big(-\lambda(t-\tau)\big)\, d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(\frac{-\tau^2}{2{\sigma_f}^2} + \lambda\tau - \lambda t\right) d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\tau^2}{2{\sigma_f}^2} - \lambda\tau\right) - \lambda t\right) d\tau$$

$$= a_f a_s \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\tau}{\sigma_f\sqrt{2}} - \frac{\sigma_f \lambda}{\sqrt{2}}\right)^2 + \frac{{\sigma_f}^2\lambda^2}{2} - \lambda t\right) d\tau$$

$$= a_f a_s \exp\left(\frac{{\sigma_f}^2\lambda^2}{2} - \lambda t\right) \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\tau}{\sigma_f\sqrt{2}} - \frac{\sigma_f \lambda}{\sqrt{2}}\right)^2\right) d\tau \quad (53)$$

The permutation integral is possible(54). Use the gaussian integral again.

$$x = \frac{\tau}{\sigma_f\sqrt{2}} - \frac{\sigma_f \lambda}{\sqrt{2}} \quad (54)$$

$$\tau = \sigma_f\sqrt{2}\left(x + \frac{\sigma_f \lambda}{\sqrt{2}}\right) = \sigma_f\sqrt{2}x + {\sigma_f}^2\lambda$$

$$\frac{d\tau}{dx} = \sigma_f\sqrt{2}$$

The observed signal $m(t)$ becomes (55).

$$m(t) = a_f a_s \exp\left(\frac{{\sigma_f}^2\lambda^2}{2} - \lambda t\right) \int_{-\infty}^{\infty} \exp(-x^2) \frac{d\tau}{dx} dx$$

$$= a_f a_s \exp\left(\frac{{\sigma_f}^2\lambda^2}{2} - \lambda t\right) \sigma_f\sqrt{2}\sqrt{\pi}$$

$$= a_f a_s \sigma_f \sqrt{2\pi} \exp\left(\frac{{\sigma_f}^2\lambda^2}{2} - \lambda t\right)$$

$$= a_f a_s \sigma_f \sqrt{2\pi} \exp\left(\frac{{\sigma_p}^2 \lambda^2}{2}\right) \exp(-\lambda t) \quad (55)$$

The result becomes the decay curve which is the exponent function $s(t)$ itself. This can be established by the semi-infinite integral interval like the gaussian function's combination(56).

$$m(t) = a_f a_s \exp\left(\frac{{\sigma_f}^2 \lambda^2}{2} - \lambda t\right) \int_0^\infty \exp(-x^2) \frac{d\tau}{dx} dx$$

$$= a_f a_s \sigma_f \sqrt{\frac{\pi}{2}} \exp\left(\frac{{\sigma_p}^2 \lambda^2}{2}\right) \exp(-\lambda t) \quad (56)$$

Although, the exponent function does not have a half width, this result gives an insight for the convolution. The difference between the exponent function and the gaussian function is the speed for their decay.

* The index shoulder of the exponent function is 1 order.
* The index shoulder of the gaussian function is 2 order.

To be natural, the exponent function is a curve which decays slower than the gaussian function($\mathbf{t > 1}$). Therefore, the convoluted waveform can be described that the curve which is slower than the other can be dominant.

Real Response Model

Up to now, the contents may be bored things by the formula derivations. But, if the observed signal was approximated(curve fitting) by the function above, it made sense. For the real system, be careful as below.

* Integral interval is limited
* There is a case that the convolution interval is a positive interval as above 0 (time series)
* Sampling by the discrete system
* Repetition system(response accumulation)

There is sometimes a case using the iteration method that the difference between the convoluted waveform and observed waveform will be minimized. But it is impractical from the point of the calculation cost. In the other hand, for the repetition system, the deconvolution is suitable. By the deconvolution of the frequency analysis(FFT), the system response function's waveform can be specified directly. The response function will be a transfer function as the spectrum. The high frequency is often observed by the spectrum analyzer, it is an observation of S parameters(spectrum). As the author's experience, the waveform's reconstruction is possible like this case. The rising and falling timing of the waveform can be observed by this way, then it becomes a great help(the author has an related patent achievement).

Conclusion

The author has been having chances to think about the system ability for a long time. From this environment, there was a suspicion scene where the system degradation is. In that scene, this paper might be of some help, the author hopes.